Lesson 8. Derivatives and Integrals of Vector Functions

0 Warm up

Example 1. Find these derivatives and integrals.

a. $\dfrac{d}{dt}(1+t^3) =$ ☐

b. $\dfrac{d}{dt}(\cos 2t) =$ ☐

c. $\dfrac{d}{dt}(te^{-t}) =$ ☐

d. $\displaystyle\int 2t\,dt =$ ☐

e. $\displaystyle\int 2\cos t\,dt =$ ☐

f. $\displaystyle\int_0^{2\pi} 3\,dt =$ ☐

1 In this lesson...

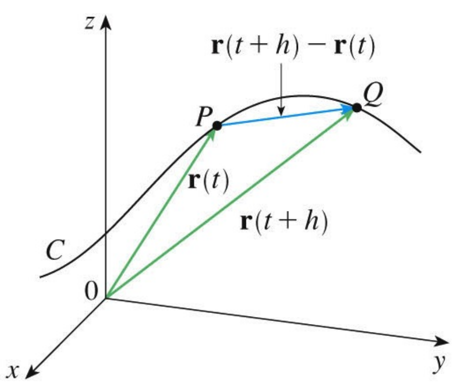
- What are the derivatives and integrals of vector functions?
- How can find the length of an arc?

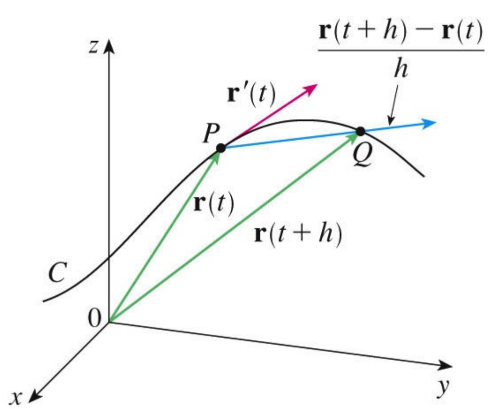
2 Derivatives

- The **derivative** of $\vec{r}$ is

$$\frac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

- Note: the derivative of a vector function <u>is also a vector</u>





- Let C be the curve defined by $\vec{r}$
- Let $\vec{r}(t)$ be the position vector of P
- The derivative $\vec{r}'(t)$ is the direction vector of the line <u>tangent</u> to C at P

  ⇒ Sometimes we refer to $\vec{r}'(t)$ as the **tangent vector**

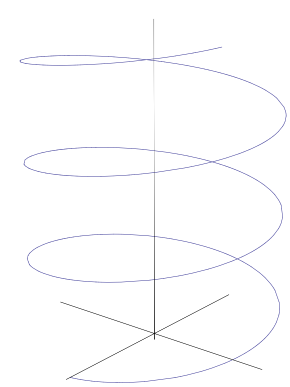
- The **unit tangent vector** is

- If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f, g, and h are differentiable functions, then

Example 2.

a. Find the derivative of $\vec{r}(t) = \langle \cos 2t, 1 + t^3, te^{-t} \rangle$.
b. Find the unit tangent vector at the point where $t = 0$.

Example 3. Find parametric equations for the tangent line to the curve given by $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ at point $(0, 1, \pi/2)$.



- Differentiation rules:

$$\frac{d}{dt}(\vec{u}(t)+\vec{v}(t)) = \vec{u}'(t)+\vec{v}'(t) \qquad \frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}'(t)\cdot\vec{v}(t)+\vec{u}(t)\cdot\vec{v}'(t)$$

$$\frac{d}{dt}(c\vec{u}(t)) = c\vec{u}'(t) \qquad \frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \vec{u}'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}'(t)$$

$$\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t)+f(t)\vec{u}'(t) \qquad \frac{d}{dt}(\vec{u}(f(t)) = f'(t)\vec{u}'(f(t))$$

3 Integration

- Let f, g, and h be continuous functions
- The **indefinite integral** of a vector function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is

- The **definite integral** of a vector function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ from a to b is

- Note: the integral of a vector function is also a vector

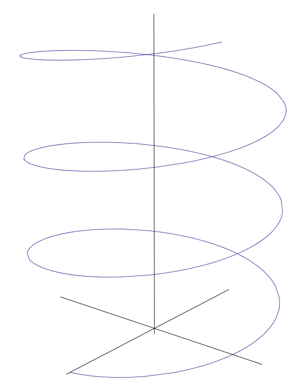
Example 4. Let $\vec{r}(t) = \langle 2\sin t, 2\cos t, 2t\rangle$. Find $\int_0^{\pi/2} \vec{r}(t)\, dt$.

4 Arc length

- Let C be a curve with vector equation $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$
- What is the length of C?
- The **arc length** of C is

- Similar for curves in 2D

Example 5. Let C be the curve defined by $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$. Find the length of C.



Example 6. Let C be the curve defined by $\vec{r}(t) = \langle 1, t^2, t^3 \rangle$, $0 \le t \le 1$. Find the length of C.